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INNER PRODUCT SPACES ASSOCIATED WITH POINCARÉ COMPLEXES

BY

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ABSTRACT. We consider the homotopy type classification of a certain kind of Poincaré complex. First we define an inner product space associated with such a Poincaré complex and we investigate the relation between the inner product space and the homotopy type of the Poincaré complex. As an application, some results for manifolds are proved.

0. Introduction. Let M be an oriented closed manifold of dimension $2n$. It is well known that the bilinear map

$$H^n(M; Z) \times H^n(M; Z) \to Z, \qquad ((x, y) \to \langle x \cup y, \mu_M \rangle)$$

makes $H^n(M; Z)$ an inner product space (in the sense of [**2**]) if M has no torsion. This inner product space is closely related with the homotopy type of M. For example, J. Milnor has proved in [**2**, p. 103] the following (cohomology version).

THEOREM. *Let M_i $(i = 1, 2,)$ be simply connected closed manifolds of dimension* 4. *Their inner product spaces are isomorphic to each other if and only if they have the same oriented homotopy type.*

We are interested in a generalization of Milnor's theorem to the case of torsion. In §1 we shall define an inner product space over Q/Z associated with an oriented Poincaré complex and in §§2–4 apply this to the homotopy type classification of Poincaré complexes K which satisfy the conditions

(1) K is $(n - 1)$-connected and of dimension $2n + 1$,

(2) $H_n(K; Z)$ is a finite abelian group G without 2-torsion $(n \geqslant 3)$.

We call K such as above a Poincaré complex of type $P^n(G)$ and in §2 we shall discuss a special case $G = \Sigma^s Z_{p^i}$. In §3 we shall give a decomposition theorem.

THEOREM A. *Let K be a Poincaré complex of type $P^n(G)$ and let $G = \sum G_p^i$ be a direct sum decomposition of G, where G_p^i is the p^i-component of G. Then K has the same oriented homotopy type as the connected sum of Poincaré complexes of type $P^n(G_p^i)$ $(i = 1, 2, \ldots,$ odd primes $p)$.*

Finally, in §4 we prove

THEOREM B. *Let K and K' be S-reducible Poincaré complexes of type $P^n(G)$. Then K has the same oriented homotopy type as K' if and only if their product spaces are isomorphic.*

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As an application we also prove

THEOREM C. *Let M be a closed smooth manifold whose underlying Poincaré complex is of type $P^n(G)$. Then if $n \not\equiv 3$ mod 4 the oriented homotopy type of M is determined by the isomorphism classes of the inner product space associated with M.*

REMARK. In the case $n \equiv 3$ mod 4, Theorem C also holds for M with trivial Pontrjagin class $P_{n+1/4}(M)$.

In particular, for a manifold which is of type $P^n(G)$ we have

COROLLARY C-1. *If $n \equiv 0$ mod 2, the oriented homotopy type of M is uniquely determined by $H_n(M; Z)$.*

COROLLARY C-2. *If $n \equiv 1$ mod 4, M is the connected sum $\# M_{p^i,k}$ up to oriented homotopy equivalence, where $M_{p^i,k}$ is a closed manifold whose underlying Poincaré complex is of type $P^n(Z_{p^i})$.*

In §§1–4 we always assume that $n \geqslant 3$ and p is an odd prime number.

1. Inner product spaces. Let K be a simply connected Poincaré complex with the fundamental class $\mu_K \in H_{2n+1}(K; Z)$. Let δ: $H^*(K; Q/Z) \to H^{*+1}(K; Z)$ be the connecting homomorphism associated with the exact sequence of coefficients, $0 \to Z \to Q \to Q/Z \to 0$.

We consider the bilinear map

$$\beta_K: H^n(K; Q/Z) \times H^n(K; Q/Z) \to Q/Z$$

defined by the composite map

$$H^n(K; Q/Z) \times H^n(K; Q/Z) \xrightarrow{1\times\delta} H^n(K; Q/Z) \times H^{n+1}(K; Z)$$
$$\xrightarrow{1\times D} H^n(K; Q/Z) \times H_n(K; Z) \xrightarrow{\langle\, ,\, \rangle} Q/Z,$$

where D is the Poincaré duality map and $\langle\, ,\, \rangle$ denotes the Kronecker pairing. The following is well known,

LEMMA 1.1. $\beta_K(x, y) = (-1)^{n+1}\beta_K(y, x)$.

Moreover assuming that $H^n(K; Z)$ (and hence $H^{n+1}(K; Z)$) are torsion, we have

PROPOSITION 1.2. *β_K is a completely orthogonal pairing (cf. [2]) and symmetric, for odd n, and skew symmetric, for even n.*

PROOF. Clearly the latter follows from Lemma 1.1. The former follows from the Poincaré duality theorem for torsion groups.

Thus we have the inner product space $V(K) = \{H^n(K; Q/Z), \beta_K\}$ associated with a Poincaré complex K. Clearly this inner product space is an oriented homotopy type invariant for oriented Poincaré complexes and we are interested in the problem, "Is the converse true?" The next section investigates a special case.

2. A special case $G = \Sigma^s Z_{p^i}$. Let $M(n, p^i)$ be the Moore space of type (n, Z_{p^i}). Note the following easy

LEMMA 2.1. $\pi_{n+1}(M(n, p^i)) = 0 = \pi_{n+2}(M(n, p^i))$.

To study the homotopy type classification of Poincaré complexes which are $(n-1)$-connected and of dimension $2n+1$ having $H_n(K; Z) = \Sigma^s Z_{p^i}$, we must investigate the homotopy group $\pi_{2n}(M_s)$ where M_s denotes the wedge sum of s copies of $M(n, p^i)$. Clearly a Poincaré complex K has the same homotopy type as the mapping cone of a map $f\colon S^{2n} \to M_s$. First we note the following two lemmas which are proved by standard arguments.

LEMMA 2.2. *The smash product* $M(n,p^i) \wedge M(n,p^i)$ *is the wedge sum* $M(2n,p^i) \vee M(2n+1,p^i)$ *up to homotopy.*

LEMMA 2.3. $\pi_{2n}(M_s) = \pi_{2n}(M_{s-1}) \oplus \pi_{2n}(M_1) \oplus \pi_{2n+1}(M_{s-1} \wedge M_1)$.

Secondly we consider the special case $s = 2$. Let P be the natural map $M(n,p^i) \to S^{n+1} = M(n,p^i)/S^n$ and consider the map

$$\hat{P} = \mathrm{id} \vee P\colon M(n,p^i) \vee M(n,p^i) \to M(n,p^i) \vee S^{n+1}.$$

Then we have the commutative diagram,

$$\begin{array}{ccccccc} \pi_{2n}(M_2) & = \pi_{2n}(M_1) & \oplus & \pi_{2n}(M_1) & \oplus & & \pi_{2n+1}(M_1 \times M_1, M_2) \\ \downarrow \hat{P}_* & \searrow \mathrm{id} & & \searrow P_* & & & \searrow (\mathrm{id} \times P)_* \\ \pi_{2n}(M_1 \vee S^{n+1}) & = \pi_{2n}(M_1) & \oplus & \pi_{2n}(S^{n+1}) & \oplus & & \pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}), \end{array}$$

in which $(\mathrm{id} \times P)_*$ is isomorphic because we have isomorphisms

$$\begin{array}{ccc} \pi_{2n+1}(M_1 \times M_1, M_1 \vee M_1) & \overset{\approx}{\rightarrow} & H_{2n+1}(M_1 \times M_1, M_1 \vee M_1) \\ \downarrow(\mathrm{id} \times P)_* & & \downarrow \approx \\ \pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}) & \overset{\approx}{\rightarrow} & H_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}). \end{array}$$

Let ι be the inclusion

$$S^n \hookrightarrow M_1 = M(n,p^i) = S^n \cup_{p^i} e^{n+1}.$$

Since it is easily seen that the Whitehead product $[\iota, \iota_{n+1}]$ generates the group $\pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1})$, we can take the element $\alpha = \hat{P}_*^{-1}([\iota, \iota_{n+1}])$ as a generator for the group $\partial\pi_{2n+1}(M_1 \times M_1, M_2)$ using the above diagram. Let i_k be the inclusion of M_1 into the kth factor of M_s and $i_{k,l}$ be the map $M_2 = M_1 \vee M_1 \to M_s$ ($i_{k,l} = i_k \vee i_l$). Then by Lemmas 2.2 and 2.3, we have

PROPOSITION 2.4. $\pi_{2n}(M_s) = \Sigma^s \pi_{2n}(M_1) \oplus \Sigma Z_{p^i}[\alpha_{k,l}]$, *where the second summation runs over* $1 \leqslant k < l \leqslant s$ *and* $\alpha_{k,l} = i_{k,l*}(\alpha)$.

Let x_0 be the fixed generator of $H^n(M(n,p^i); Q/Z) \approx Z_{p^i}$ and let p_k be the projection $M_s \to M_1$ onto the kth factor of M_s. We take $\{x_k = p_k^*(x_0)\}$ as a system of generators for $H^n(M_s; Q/Z)$ and identify $H^n(M_s; Q/Z)$ with $H^n(K_f; Q/Z)$ for the mapping cone K_f of a map $f\colon S^{2n} \to M_s$. Now we define a homomorphism

$$D_s\colon \pi_{2n}(M_s) \to \{s \times s\text{-matrices over } Q/Z\}$$

by the formula

$$D_s(f) = (f_{k,l}), \qquad f_{k,l} = \langle x_k, \mu_f \cap \delta x_l \rangle,$$

where μ_f denotes the oriented generator of $H_{2n+1}(K_f; Z)$.

LEMMA 2.5. *The matrix* $(f_{k,l})$ *is symmetric for odd* n *and skew symmetric for even* n. *Moreover,* K_f *is a Poincaré complex if and only if* $D_s(f) = (f_{k,l})$ *is invertible.*

PROOF. The first part follows from Lemma 1.1. The second part is equivalent to Poincaré duality.

Let φ be a map $M_s \to M_t$ and define the matrix $U = (u_{k,l})$ $(u_{k,l} \in Q/Z)$ by $\varphi^*(x_k) = \Sigma_l u_{k,l} x_l$ $(k = 1, \dots, t)$. Then by the definition of D_s and U, we have

LEMMA 2.6 (NATURALITY OF D_s). *For a map* φ: $M_s \to M_t$, *we have*

$$D_t\big(\varphi_*(f)\big) = U \circ D_s(f) \circ {}^tU \qquad (f \in \pi_{2n}(M_s)).$$

Now we prove a key lemma for our purpose.

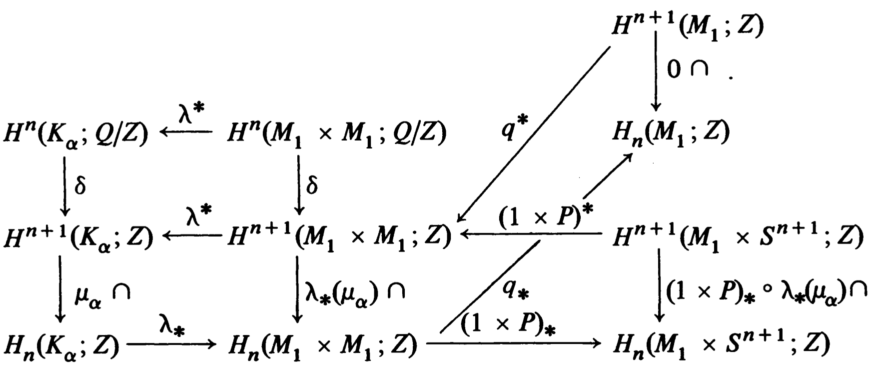
LEMMA 2.7. *Let* $f = \Sigma f_k + \Sigma a_{k,l}[\alpha_{k,l}]$ $(k < l)$ *be the decomposition in Proposition* 2.4. *Then we have*

$$f_{k,l} = a_{k,l} \quad (k < l) \quad \text{and} \quad f_{k,k} = D_1(f_k')$$

where $D_s(f) = (f_{k,l})$ *and* $i_{k*}(f_k') = f_k$.

PROOF. The equation $f_{k,k} = D_1(f_k')$ follows from the naturality for the projection p_k: $M_s \to M_1$ $(p_k(f) = f_k')$. For $f_{k,l} = a_{k,l}$, it is sufficient to prove $D_s(\alpha_{k,l}) = \{(k, l)$ and (l, k) components are ± 1 and others are $0\}$, and this is equivalent to $D_2(\alpha) = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ by the naturality. And so we consider as follows:

Now there exists a map λ: $K_\alpha \to M_1 \times M_1$ such that $\lambda | M_2 = \mathrm{id}_{M_2}$: $M_2 \to M_2 \subset M_1 \times M_1$ and $\lambda_*(\mu_\alpha)$ is a generator of $H_{2n+1}(M_1 \times M_1; Z) \approx Z_{p^i}$. Then we consider the following diagram:



where P denotes the natural collapsing map $M_1 \to M_1/S_n$ and q denotes the projection $M_1 \times M_1 \to M_1$ onto the first factor. Then it is easily verified that $D_2(\alpha) = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ using the standard generators of $H_n(K_\alpha; Z)$ and $H^{n+1}(K_\alpha; Z)$. Thus the proof of the lemma is completed.

Here we recall that for two matrices A and B over a ring R $A \equiv B$ if and only if there exists an invertible matrix U such that $U \circ A \circ {}^tU = B$. Then the preceding argument shows

LEMMA 2.8. $D_s(f) \equiv D_s(g)$ *for* $f, g \in \pi_{2n}(M_s)$ *if and only if there exists a homotopy equivalence* φ: $M_s \to M_s$ *such that* $\varphi_*(f) \equiv g \bmod D_s^{-1}(0)$.

Let K_i ($i = 1, 2$) be two Poincaré complexes of type $P^n(\Sigma^s Z_{p^i})$ and we shall prove the main proposition in this section.

PROPOSITION 2.9. *If K_i are both S-reducible, then K_1 has the same homotopy type as K_2 if and only if $V(K_1)$ is isomorphic to $V(K_2)$.*

Let

$$E^N\colon \pi_{2n}(M_s) \to \pi_{2n+N}(E^N M_s) \qquad (N \to \infty)$$

denote the suspension homomorphism. Specially we use the restricted homomorphism

$$E^N\colon D_s^{-1}(0) \to \pi_{2n+N}(E^N M_s).$$

Since $D_s^{-1}(0)$ is decomposed into the direct sum $\Sigma i_{k*} D_1^{-1}(0)$ by Proposition 2.4, it is sufficient for us to consider the case $s = 1$. Then from Diagram 1.6 of [4] (h_n is essentially D_1), we get

LEMMA 2.10. *$E^N\colon D_1^{-1}(0) \to \pi_{2n+N}(E^N M_1)$ is injective.*

Now we prove Proposition 2.9. Since K_1 has the same homotopy type as the mapping cone K_{f_1} for a map $f_1\colon S^{2n} \to M_s$, the inner product space $V(K_1)$ is isomorphic to $V(K_{f_1})$. If $V(K_{f_1})$ is isomorphic to $V(K_{f_2})$ for another map $f_2\colon S^{2n} \to M_s$, there exists a homotopy equivalence $\varphi\colon M_s \to M_s$ such that $\varphi_*(f_1) \equiv f_2 \bmod D_s^{-1}(0)$, by Lemma 2.8. Since the S-reducibility of K_i means $E^N(f_i) = 0$ ($N \to \infty$), the proof is completed from Lemma 2.10.

COROLLARY 2.11. *Let K be a Poincaré complex of type $P^n(\Sigma^s Z_{p^i})$. Then if n is odd, K is the connected sum of Poincaré complexes of type $P^n(Z_{p^i})$ up to homotopy.*

COROLLARY 2.12. *If n is even, the homotopy type for S-reducible Poincaré complex of type $P^n(\Sigma^s Z_{p^i})$ is uniquely determined by s.*

PROOF. K is the mapping cone K_f up to homotopy and we may assume that $D_s(f)$ is a diagonal matrix if n is odd. Hence f has the decomposition $f = \Sigma f_k$ in Proposition 2.4. This means that K_f is the connected sum $\#^s K_{f_k}$ up to homotopy. Since K_{f_k} is a Poincaré complex of type $P^n(Z_{p^i})$, the proof of Corollary 2.11 is completed. Moreover since we know that $D_s(f)$ is unique up to equivalence if n is even, Corollary 2.12 follows from Proposition 2.9.

3. Decomposition (up to homotopy). First we define a generalized Hopf homomorphism $H_G\colon \pi_{2n}(M(n, G)) \to \operatorname{End} G$ by the formula

$$H_G(f) = \mu_f \cap : H^{n+1}(K_f; Z) = G \to H_n(K_f; Z) = G$$

where $\mu_f \in H_{2n+1}(K_f; Z)$ denotes the oriented generator of the mapping cone K_f for a map $f\colon S^{2n} \to M(n, G)$. Let $\rho\colon G \to H$ be a homomorphism and $\hat{\rho}\colon M(n, G) \to M(n, H)$ be the map induced by ρ ($\rho = \hat{\rho}_*\colon H_n(M(n, G); Z) = G \to H_n(M(n, H); Z) = H$). If we denote by ${}^t\rho$ the homomorphism

$$\rho^*\colon H^{n+1}(M(n, H); Z) = H \to H^{n+1}(M(n, G); Z) = G,$$

then we have

LEMMA 3.1. *$H_H(\hat\rho \circ f) = \rho \circ H_G(f) \circ {}^t\rho$ for $f \in \pi_{2n}(M(n, G))$.*

PROOF. Let $\hat\rho_f: K_f \to K_{\hat\rho \circ f}$ be the natural extension of $\hat\rho$ and consider the diagram

$$\begin{array}{ccc} H^{n+1}(K_f; Z) = G & \underset{\mu_f \cap}{\rightarrow} & H_n(K_f; Z) = G \\ \uparrow \rho_f^* & & \downarrow \rho_{f*} \\ H^{n+1}\left(K_{\hat\rho \circ f}; Z\right) = H & \underset{\mu_{\hat\rho \circ f} \cap}{\rightarrow} & H_n\left(K_{\hat\rho \circ f}; Z\right) = H \end{array}$$

Then the proof follows from the commutativity of the diagram.

Let $G = G_1 \oplus G_2$ be a direct sum decomposition of G. From Lemma 3.1 we have the diagram

$$\begin{array}{ccccccc} \pi_{2n}(M(n, G)) = & \pi_{2n}(M(n, G_1)) & \oplus & \pi_{2n}(M(n, G_2)) & \oplus & \pi_{2n+1}(M(n, G_1) \times M(n, G_2), \vee) & \\ \downarrow H_G & \downarrow H_{G_1} & & \downarrow H_{G_2} & & \swarrow\searrow & \\ \text{End } G = & \text{End } G_1 & \oplus & \text{End } G_2 & \oplus & \text{Hom}(G_1, G_2) \oplus \text{Hom}(G_2, G_1). & \end{array} \tag{3.2}$$

LEMMA 3.3. *The restriction $H_G|\partial\pi_{2n+1}(M(n, G_1) \times M(n, G_2), \vee)$ is injective.*

PROOF. Let $G_1 = \Sigma\, Z_{p^i}[x_{p,i,l}]$ and $G_2 = \Sigma\, Z_{p^j}[y_{p,j,k}]$ be a direct sum decomposition of G_1 and G_2 respectively. Since we have decompositions

$$\begin{aligned} &\pi_{2n+1}(M(n, G_1) \times M(n, G_2), M(n, G_1) \vee M(n, G_2)) \\ &\qquad = \pi_{2n+1}(M(n, Z_{p^i}) \times M(n, Z_{p^j}), M(n, Z_{p^i}) \vee M(n, Z_{p^j})) \end{aligned}$$

and $\text{Hom}(G_1, G_2) \oplus \text{Hom}(G_2, G_1) = \Sigma\, \text{Hom}(Z_{p^i}, Z_{p^j}) \oplus \text{Hom}(Z_{p^j}, Z_{p^i})$, it is sufficient for us to consider the case $G_1 = Z_{p^i}[x]$ and $G_2 = Z_{p^j}[y]$. Thus the proof follows by an argument similar to the proof of Lemma 2.7.

REMARK. We note that the proof also shows $H_G(f_3)|G_2 = {}^t(H_G(f_3)|G_1)$ for $f = f_1 + f_2 + f_3$ in diagram (3.2).

LEMMA 3.4. *Let $f = f_1 + f_2 + f_3$ be the decomposition of f in diagram* (3.2). *If $H_{G_1}(f_1)$ is contained in* Aut G_1, *then there exists a homotopy equivalence $h: M(n, G) \to M(n, G)$ such that $h_*(f)$ is contained in $\pi_{2n}(G_1) \oplus \pi_{2n}(G_2)$ and $H_{G_1}(f_1) = H_{G_1}(f_1')$ where $h_*(f) = f_1' + f_2'$.*

PROOF. If we show that there exists an isomorphism $\lambda: G \to G$ such that $\lambda \circ H_G(f) \circ {}^t\lambda|\text{End } G_1 = 0$ and $\lambda \circ H_G(f) \circ {}^t\lambda|\text{Hom}(G_1, G_2) = H_{G_1}(f_1)$, then the proof follows from Lemmas 3.1, 3.3 and the above remark.

For homomorphisms $\lambda_{i,j}: G_i \to G_j$ $(i, j = 1, 2)$, we define a homomorphism λ: $G \to G$ by $\lambda(g) = \lambda_{1,1}(g_1) + \lambda_{2,1}(g_2) + \lambda_{1,2}(g_1) + \lambda_{2,2}(g_2)$ for $g = g_1 + g_2$. Then from easy computation, we can get

$$\begin{aligned} \lambda \circ h_G(f) \circ {}^t\lambda|\text{End } G_1 &= \lambda_{1,1} \circ H_{G_1}(f_1) \circ {}^t\lambda_{1,1} + \lambda_{2,1} \circ H_G(f_3)|G_1 \circ {}^t\lambda_{1,1} \\ &\quad + \lambda_{1,1} \circ H_G(f_3)|G_2 \circ {}^t\lambda_{2,1} + \lambda_{2,1} \circ H_{G_2}(f_2) \circ {}^t\lambda_{2,1} \end{aligned}$$

and

$$\lambda \circ H_G(f) \circ {}^t\lambda | \mathrm{Hom}(G_1, G_2) = \lambda_{1,2} \circ H_{G_L}(f_1) \circ {}^t\lambda_{1,1} + \lambda_{2,2} \circ H_G(f_3)|G_1 \circ {}^t\lambda_{1,1} + \lambda_{1,2} \circ H_G(f_3)|G_2 \circ {}^t\lambda_{2,1} + \lambda_{2,2} \circ H_{G_2}(f_2) \circ {}^t\lambda_{2,1}.$$

Hence if we take $\lambda_{1,1} = \mathrm{id}_{G_1}$, $\lambda_{2,2} = \mathrm{id}_{G_2}$, $\lambda_{2,1} = 0$ and

$$\lambda_{1,2} = -H_G(f_3)|G_1 \circ H_{G_1}(f_1)^{-1},$$

then λ satisfies our conditions.

LEMMA 3.5. *Suppose $G = G_1 \oplus G_2$ where $G_1 = \sum Z_{p^m}$ and $p^{m-1}G_2$ has a trivial p-primary component. If $H_G(f)$ is an automorphism for $f = f_1 + f_2 + f_3$, then $H_{G_1}(f_1)$ is also an automorphism.*

PROOF. Consider the following exact ladder

$$\begin{array}{ccccccccc}
0 \longrightarrow & \ker p^{m-1} & \longrightarrow & G_1 & \xrightarrow{p^{m-1}} & G_1 & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow 0 \\
& \downarrow \approx & & \downarrow i_1 & & \downarrow i_1 & & \downarrow \approx & \\
0 \longrightarrow & \ker p^{m-1} & \longrightarrow & G & \xrightarrow{p^{m-1}} & G & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow 0 \\
& \downarrow & & \downarrow H_G(f) & & \downarrow H_G(f) & & \downarrow & \\
0 \longrightarrow & \ker p^{m-1} & \longrightarrow & G & \xrightarrow{p^{m-1}} & G & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow 0 \\
& \downarrow \approx & & \downarrow p_1 & & \downarrow p_1 & & \downarrow \approx & \\
0 \longrightarrow & \ker p^{m-1} & \longrightarrow & G_1 & \longrightarrow & G_1 & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow 0
\end{array}$$

where i_1 is the inclusion $G_1 \hookrightarrow G$ and p_1 is the projection $G \to G_1$ and p^{m-1} denotes the map multiplied by p^{m-1}. The right and left vertical maps are isomorphisms because $H_G(f)$ is also. Hence it follows that $H_{G_1}(f_1) = p_1 \circ H_G(f) \circ i_1$ is surjective and so an isomorphism. Now Theorem A follows from Lemmas 3.4 and 3.5.

4. Proofs of Theorems B and C.

LEMMA 4.1. Let K_i $(i = 1, 2)$ *be two Poincaré complexes of dimension m. Then the connected sum $K_1 \# K_2$ is S-reducible if and only if K_1 and K_2 are both S-reducible.*

PROOF. This is clear because $K_1 \# K_2$ has a CW-decomposition $(K_1^{(m-1)} \vee K_2^{(m-1)}) \cup_f e^m$, where $f = f_1 + f_2$ for $K_i = K_i^{(m-1)} \cup_{f_i} e^m$.

Now we prove Theorem B. Let K and K' be two Poincaré complexes of type $P^n(G)$. Trivially $V(K) \equiv V(K')$ if K and K' have the same oriented homotopy type. Assume that $V(K) \equiv V(K')$. By Theorem A, we may assume that K and K' are the connected sums $\# K_{p,i}$ and $\# K'_{p,i}$ (corresponding to the Z_{p^i}-component of G) respectively. Let h_i: $K \to K_{p,i}$ and h'_i: $K' \to K'_{p,i}$ be the projection onto each factor. Since we may consider $\beta_K = \sum \beta_K | \sum Z_{p^i}$ and $\beta_{K'} = \sum \beta_{K'} | \sum Z_{p^i}$ by Lemmas 3.4 and 3.5, h_i (h'_i) induces an isomorphism $V(K)|\sum Z_{p^i} \cong V(K_{p,i})$ $(V(K')|\sum Z_{p^i} \cong V(K'_{p,i}))$ and so $V(K_{p,i})$ is isomorphic to $V(K'_{p,i})$. Thus $K_{p,i}$ has the same oriented homotopy type as $K'_{p,i}$ by Proposition 2.9 and Lemma 4.1, and hence K has the same oriented homotopy type as K'. Thus the proof is completed.

From Theorem A and Corollaries 2.11 and 2.12, we have

COROLLARY B-1. *Let K be a Poincaré complex of type $P^n(G)$. Then K has the same homotopy type as the connected sum of Poincaré complexes of type $P^n(Z_{p^i})$ for various p and i if n is odd.*

COROLLARY B-2. *If n is even, the oriented homotopy type for S-reducible Poincaré complexes of type $P^n(G)$ is uniquely determined by n and G.*

Let M be a closed smooth manifold of dimension $2n + 1$ whose underlying Poincaré complex is of type $P^n(G)$ and let ν_M: $M \to BO$ be the stable normal bundle for M. Since M has the same homotopy type as the mapping cone K_f for a map f: $S^{2n} \to M(n, G)$, we may replace M with K_f.

LEMMA 4.2. *If $n \not\equiv 3 \bmod 4$ then ν_M: $K_f \to BO$ is trivial.*

PROOF. First consider the restriction $\nu_M | M_1$,

$$K_f \supset M(n, G) \supset M_1 = S^n \underset{p^i}{\cup} e^{n+1} \to BO.$$

From Puppe's sequence and $\pi_n(BO) = 0$, Z or Z_2, we obtain that $\nu_M | M_1$ is trivial, and so $\nu_M | M(n, G)$ is trivial. Hence ν_M is decomposed as follows

$$\nu_M \colon K_f \xrightarrow{P} S^{2n+1} \xrightarrow{\nu} BO$$

where P denotes the natural map $K_f \to K_f / M(n, G) = S^{2n+1}$. Since P induces the map T_P: $T(\nu_M) \to T(\nu)$ of degree 1, $T(\nu)$ is reducible and ν is fiber homotopically trivial, that is, $J(\nu) = 0$ in this case. Hence ν is trivial by Adams' Theorem. This completes the proof.

Thus since Lemma 4.2 shows that M is S-reducible, Theorem C and Corollary C-1 follow from Theorem B and Corollary B-2 respectively. Corollary C-2 is also obtained from Lemmas 4.1 and 4.2 and Corollary B-1.

For the case $n \equiv 3 \bmod 4$, the situation is more complicated and we could not obtain simple theorems. As an example we list

PROPOSITION 4.3. *Suppose* (order G, order $J(S^{n+1})) = 1$. *For any element x of $H^{n+1}(M(n, G);\ Z) \approx G$, there exists a closed smooth manifold M^{2n+1} with $P_{n+1/4}(M) = x$ which is of type $P^n(G)$ as a Poincaré complex and is a π-manifold up to homotopy.*

PROOF. The proof follows from the case $G = Z_{p^i}$, using the connected sum operation. On the other hand, the condition (p, order $J(S^{n+1})) = 1$ means that $J(K_f) = 0$ for any $f \in \pi_{2n}(M(n, Z_{p^i}))$. Since we can take f such that $D_1(f) = 1$ and $E^N(f) = 0$, the proof is completed.

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